Origin of Intermediate Spectral Properties in Quantum Chaos:

Degree of Chaos and Missing Levels

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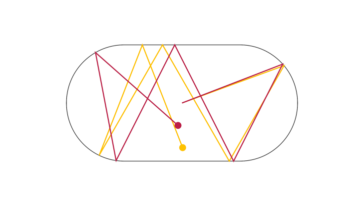
Summary

- Introduction: Classical/Quantum Chaos
- Random Matrix Theory
- Spectral fluctuations: NNSD
- Intermediate spectral properties (dynamics/missing levels)
- 2-parameter NNSD
- Tests and fits
- Conclusions

Quantum Chaos

- The concept of chaos in Classical Mechanics can not be easily carried to Quantum Mechanics



Integrability - *Butterfly effect* - *Lyapunov exponent...*

- **"Quantum Chaos" $\longrightarrow$ Classical Chaos**

Quantum Chaos

- The **spectral fluctuations** are the key property characterizing order and chaos in quantum systems

Classical system		Spectral fluctuations
Integrable	$\longrightarrow$	Poisson (Berry and Tabor) *Proc. R. Soc. Lond. A* **356**, 375 (1977)
Chaotic	$\longrightarrow$	GOE (Bohigas, Gianonni, and Schmit) *Phys. Rev. Lett.* **52**, 1 (1984)

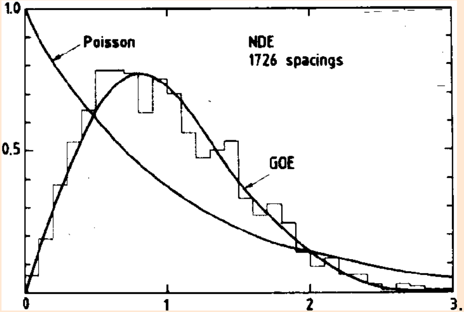
Quantum Chaos

Quantum Chaos

Random Matrix Theory (RMT)

- ORIGIN: Analysis of spectra of complex nuclei

$P(s)$



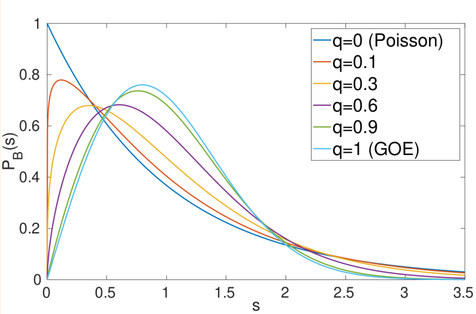
s

Haq, Pandey, and Bohigas, *Phys. Rev. Lett.* **48**, 1086 (1982)

Spectral statistics: NNSD

- Nearest neighbor spacing distribution

$$s_i = \varepsilon_{i+1} - \varepsilon_i : P(s)$$
$$\{E_i\} \to \{\varepsilon_i\} \, (\langle s \rangle = 1)$$



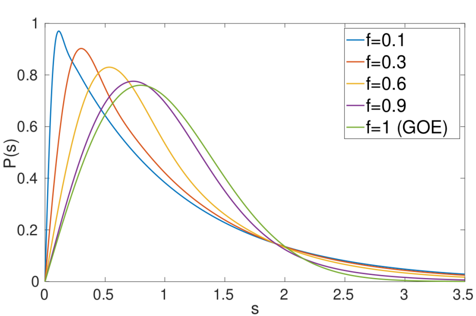
- $P_{GOE}(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right)$
- $P_{Poisson}(s) = \exp(-s)$
- $P_{Brody}(s) = a s^q \exp(-b s^{q+1})$

$$\left(a = (q+1)b,\ b = \left[\Gamma\left(\tfrac{q+2}{q+1}\right)\right]^{q+1} \right)$$

Spectral statistics: NNSD

- Intermediate spectral behavior can also be due to:

missing levels...



- $p(s) = \sum_{k=0}^{\infty}(1-f)^k p(k, s/f)$

  Bohigas and Pato, *Phys. Rev. Lett.* **595**, 171 (2004)

Aim

Find the real definition of Quantum Chaos

Aim

Distinguish the different causes of intermediate behavior in a spectrum (true **mixed dynamics** / **missing levels**) and **quantify** them

The 2-parameter formula

$$
\begin{aligned}
P_{q,f}(s) = {} & a\left(\frac{s}{f}\right)^q \exp\left[-b\left(\frac{s}{f}\right)^{q+1}\right] \\
& + (1-f)aa_1\int_0^{\frac{s}{f}}\left(\frac{s}{f}\right)^{2q} x^q \exp\left\{-bx^{q+1} - b_1\left[\left(\frac{s}{f}\right)^{2q+1} - x^{2q+1}\right]\right\} dx \\
& + (1-f)^2 aa_1a_2\int_0^{\frac{s}{f}}\int_0^x \left(\frac{s}{f}\right)^{3q} x^{2q}y^q \exp\Bigg\{-by^{q+1} - b_1\left[x^{2q+1} - y^{2q+1}\right] \\
& -b_2\left[\left(\frac{s}{f}\right)^{3q+1} - x^{3q+1}\right]\Bigg\} dydx \\
& + \sum_{n\geq 3}\frac{(1-f)^n}{\sqrt{2\pi}\sigma(n,q)}\exp\left\{-\frac{\left[\left(\frac{s}{f}\right)-n-1\right]^2}{2\sigma(n,q)}\right\}
\end{aligned}
$$

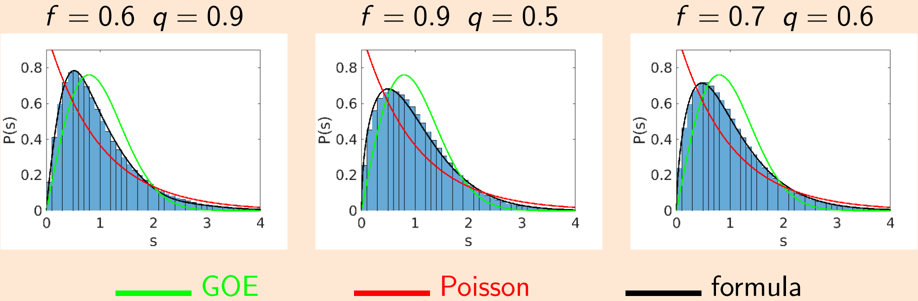
Hita-Pérez, Muñoz, and Molina, *J. Phys. A: Math. Theor.* **56**, 505702 (2023)

The 2-parameter formula

- Checking its accuracy: β-ensembles of 1000 spectra with $N = 10000$

q: **degree of chaos** (0:regular→1:chaotic)

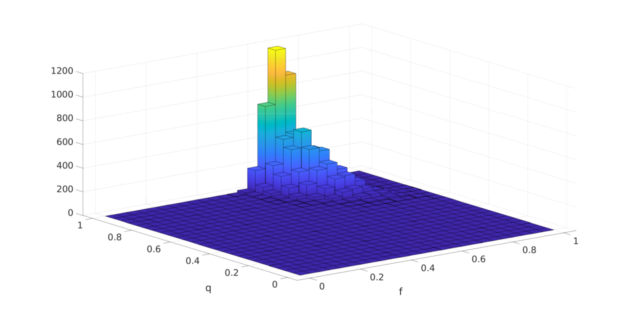
f: **fraction of observed levels**



The 2-parameter formula

- **Fitting** data (ensemble of 10000 β-matrices)

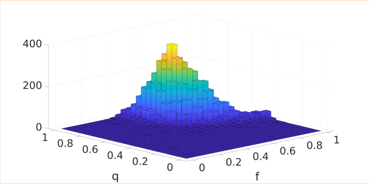
$$f = 0.8 \quad q = 0.9 \quad (N = 1000)$$



The 2-parameter formula

- **Fitting** data (ensembles of 10000 β-matrices): $f = 0.7 \;\; q = 0.6$

$N = 1000$



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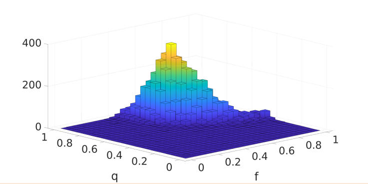
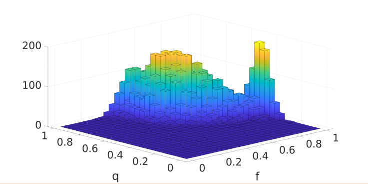


$N = 500$



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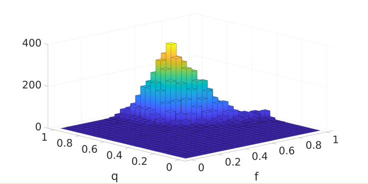

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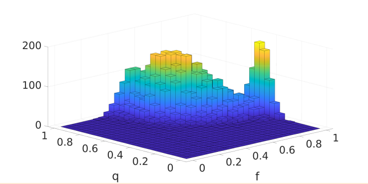
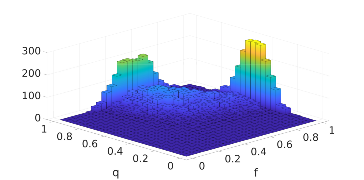


$N = 200$



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Conclusions

Aim: **Distinguish** the different causes of intermediate behavior in a spectrum (true **mixed dynamics** / **missing levels**) and **quantify** them?

We have started to answer this question:

- $P_{q,f}(s)$: 2-parameter formula for the 1st time:
  - Properly describes the transitions
    - $q : 0 \rightarrow 1$ (**degree of chaos**)
    - $f : 0 \rightarrow 1$ (**missing levels**)
  - Suitable for fitting purposes ($f \gtrsim 0.6$, $N \gtrsim 1000$, but with some previous information on the system... shorten ranges? (better fits))
- We make available the 2-parameter formula

  https://doi.org/10.5281/zenodo.8434033

  and await for **experimental**/numerical/theoretical results to continue improving it...

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Additional: Gaussian Orthogonal Ensemble (GOE)

$$M_{GOE} = \begin{pmatrix} N(0,\sigma) & N(0,\sigma/\sqrt{2}) & \cdots & N(0,\sigma/\sqrt{2}) \\ N(0,\sigma/\sqrt{2}) & N(0,\sigma) & \ddots & \vdots \\ \vdots & \ddots & \ddots & N(0,\sigma/\sqrt{2}) \\ N(0,\sigma/\sqrt{2}) & \cdots & N(0,\sigma/\sqrt{2}) & N(0,\sigma) \end{pmatrix}$$

Real symmetric matrices (invariant under orthogonal transformations)
H_{ij} are independent random Gaussian variables $N(\mu,\sigma)$, $\sigma_{ij} = \sigma ii/\sqrt{2}$

Additional: β-ensembles

$$H_\beta = \frac{1}{\sqrt{2}} \begin{pmatrix} N(0,2) & \chi_{(N-1)\beta} & & & \\ \chi_{(N-1)\beta} & N(0,2) & \chi_{(N-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & \chi_\beta & N(0,2) \end{pmatrix}$$

$\beta = 0$ (Poisson-regular) $\longrightarrow$ $\beta = 1$ (GOE-chaotic)

Dumitriu and Edelman *J. Math. Phys.* **47**, 063302 (2006)

Additional: Unfolding

- Level density: $g(E) = \overline{g}(E) + \widetilde{g}(E)$
- Accumulated level density: $m(E) = \int_{-\infty}^{E} g(x)dx = \overline{m}(E) + \widetilde{m}(E)$
- Unfolding:

$$\begin{aligned} \{E_i\} &\longrightarrow \{\varepsilon_i = \overline{m}(E_i)\} \\ \overline{g}(E) &\longrightarrow \overline{\rho}(\varepsilon) = 1 \\ \overline{m}(E) &\longrightarrow \overline{\mu}(\varepsilon) = \varepsilon \end{aligned}$$